Matrix Algebra

HGEN619 class 2005

Heuristic

- You already know a lot of it
- Economical and aesthetic
- Great for statistics

What you know

- All about (1x1) matrices

Operation	Example	Result
Addition	2 + 2	
Subtraction	5 - 1	
Multiplication	2 x 2	
Division	12 / 3	

What you know

- All about (1x1) matrices

Operation	Example	Result
Addition	2 + 2	4
Subtraction	5 – 1	4
Multiplication	2 x 2	4
Division	12 / 3	4

What you may guess

- Numbers can be organized in boxes, e.g.

$$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$$

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- Numbers can be organized in boxes, e.g.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Matrix Notation

$$\underline{A}$$

Many Numbers

31 23 16 99 08 12 14 73 85 98 33 94 12 75 02 57 92 75 11
28 39 57 17 38 18 38 65 10 73 16 73 77 63 18 56 18 57 02
74 82 20 10 75 84 19 47 14 11 84 08 47 57 58 49 48 28 42
88 84 47 48 43 05 61 75 98 47 32 98 15 49 01 38 65 81 68
43 17 65 21 79 43 17 59 41 37 59 43 17 97 65 41 35 54 44
75 49 03 86 93 41 76 73 19 57 75 49 27 59 34 27 59 34 82
43 19 74 32 17 43 92 65 94 13 75 93 41 65 99 13 47 56 34
75 83 47 48 73 98 47 39 28 17 49 03 63 91 40 35 42 12 54
31 87 49 75 48 91 37 59 13 48 75 94 13 75 45 43 54 32 53
75 48 90 37 59 37 59 43 75 90 33 57 75 89 43 67 74 73 10
34 92 76 90 34 17 34 82 75 98 34 27 69 31 75 93 45 48 37
13 59 84 76 59 13 47 69 43 17 91 34 75 93 41 75 90 74 17
34 15 74 91 35 79 57 42 39 57 49 02 35 74 23 57 75 11 35

Matrix Notation

$$\underline{A}$$

Useful Subnotation

$$\underset{2 \qquad 2}{\underline{A}}$$

Useful Subnotation

$$\underset{8 \times 40}{\underline{A}}$$

8 40

Matrix Operations

- Addition
- Subtraction
- Multiplication
- Inverse

Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underline{A} + \underline{B} =$$

Addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$\underline{A} + \underline{B} = \underline{C}$$

Addition Conformability

To add two matrices A and B:

- # of rows in A = # of rows in B
- # of columns in A = # of columns in B

Subtraction

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underline{B} - \underline{A} =$$

Subtraction

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\underline{B} - \underline{A} = \underline{C}$$

Subtraction Conformability

- To subtract two matrices A and B:
- # of rows in A = # of rows in B
- # of columns in A = # of columns in B

Multiplication Conformability

- Regular Multiplication
- To multiply two matrices A and B:
- # of columns in A = # of rows in B
- Multiply: A (m x n) by B (n by p)

Multiplication General Formula

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$$

Multiplication I

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} x \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underline{A} \quad x \quad \underline{B} \quad =$$

Multiplication II

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} x \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1) & \\ & \end{bmatrix}$$

$$\underline{A} \quad x \quad \underline{B} \quad = \quad \underline{C}$$

$$C_{11} = \sum_{k=1}^{n} A_{11} \times B_{11}$$

Multiplication III

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5x1)+(6x3) & \\ & \end{bmatrix}$$

$$\underline{A} \quad x \quad \underline{B} \quad = \quad \underline{C}$$

$$C_{11} = \sum_{k=2}^{n} A_{12} \times B_{21}$$

Multiplication IV

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & (5x2)+(6x4) \\ & \end{bmatrix}$$

$$\underline{A} \quad x \quad \underline{B} \quad = \quad \underline{C}$$

$$C_{12} = \sum_{k=1}^{n} A_{1k} \times B_{k2}$$

Multiplication V

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ (7x1)+(8x3) & \end{bmatrix}$$

$$\underline{A} \quad x \quad \underline{B} \quad = \quad \underline{C}$$

$$C_{21} = \sum_{k=1}^{n} A_{2k} \text{ x } B_{k1}$$

Multiplication VI

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & (7x2)+(8x4) \end{bmatrix}$$

$$\underline{A} \quad x \quad \underline{B} \quad = \quad \underline{C}$$

$$C_{22} = \sum_{k=1}^{n} A_{2k} \text{ x } B_{k2}$$

Multiplication VII

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$\underset{m \times n}{\underline{A}} \times \underset{n \times p}{\underline{B}} = \underset{m \times p}{\underline{C}}$$

Inner Product of a Vector

- (Column) Vector **c** (n x 1) $\underline{c} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2x2)+(4x4)+(1x1) \end{bmatrix} = \begin{bmatrix} 21 \end{bmatrix}$$

$$\underline{c}' \quad \underline{c} \quad = \quad \underline{c}' \, \underline{c}$$

Outer Product of a Vector

- (Column) vector **c** (n x 1) $\underline{c} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 2 \\ 8 & 16 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

$$\underline{c} \qquad \underline{c}' \qquad \underline{c}\,\underline{c}'$$

Inverse

- A number can be divided by another number - How do you divide matrices?
- Note that a / b = a x 1 / b
- And that   a x 1 / a = 1
- 1 / a is the inverse of a

Unany operations: Inverse

- Matrix 'equivalent' of 1 is the identity matrix

$$\underline{\mathbf{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Find A^{-1} such that $A^{-1} * A = I$

Unary Operations: Inverse

- Inverse of (2 x 2) matrix
  - Find determinant
  - Swap a_{11} and a_{22}
  - Change signs of a_{12} and a_{21}
  - Divide each element by determinant
  - Check by pre- or post-multiplying by inverse

Inverse of 2 x 2 matrix

- Find the determinant
  $= (a_{11} \times a_{22}) - (a_{21} \times a_{12})$
  For

$$\underline{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

det(**A**) = (2x3) – (1x5) = 1

  - A determinant is a scalar number which is calculated from a matrix. This number can determine whether a set of linear equations are solvable, in other words whether the matrix can be inverted.

Inverse of 2 x 2 matrix

- Swap elements a_{11} and a_{22}
  Thus

$$\underline{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Inverse of 2 x 2 matrix

- Change sign of a_{12} and a_{21}

Thus

$$\underline{A} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Inverse of 2 x 2 matrix

- Divide every element by the determinant

Thus

$$\underline{A} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

(luckily the determinant was 1)

Inverse of 2 x 2 matrix

- Check results with $\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$

  Thus

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

  equals

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$